Problem Solving and Communication Activity Series

Make a Mathematical Model

A mathematical model is a way to describe a situation, usually real-world, using numeric and mathematical relationships. Mathematical models usually have inputs, operations on those inputs, certain parameters or constants that make the operations fit the particular situation, and outputs that result from performing the operations on the inputs.

While students in Algebra and beyond use mathematical models with functions and variables, students of all ages describe situations mathematically. An Algebra student might mathematize the idea “apples cost fifty cents each,” as $p(a) = 0.5a$, but an elementary student would be expressing the same idea, as mathematically as possible, when they say, “divide the number of apples by 2, because it is 2 for a $1, and $.50 if one is left over.”

Sometimes in problem solving, coming up with the mathematical model to use is at the heart of the problem. Problem-solvers are engaged in noticing quantities and relationships, selecting operations to describe the relationships, and fitting those operations to the specific scenario by setting parameters.

Other times, the operations and relationships are given in the problem, and the problem solver’s job is to organize the information and apply it to determine a final answer.

In either case, identifying quantities and relationships, and recording information as mathematically as possible are key components of making mathematical models.

The activities below help students to organize and write succinctly the mathematical quantities and relationships they observe.

The activities are written so that you can use them with problems of your choosing.

Problem-Solving Goals

Making a mathematical model can help problem-solvers:

- Organize information given in a problem to streamline the problem-solving process.
- Represent the essential mathematical aspects of a system.
- Develop rules and algorithms that can be generalized to multiple situations.
- Generalize and formalize insights.

Communication Goals

Problem-solvers who are making mathematical models use the writing process to:

- Help them organize the quantities and relationships they observe.
- Label and define quantities clearly.
- Make sure their math is represented as succinctly and symbolically as is age-appropriate.

Activities

I. Organizing Quantities and Relationships

Format: students working individually or in pairs, then sharing with groups of 4-6.

Building a model, whether physical or mathematical, the first step is usually to make sure you have all the pieces and know what parts of the model they are. In this activity, the focus is on organizing and recording carefully the quantities and relationships you notice.

Sample Activity

Mathematical models pay more attention to the **quantities** and the **relationships** between them than they do to the specific **values**. Mathematical models can be built by identifying the key **quantities** in a situation and putting them together in a **relationship** using operations such as addition, subtraction, multiplication, division, etc.

Work individually or in pairs on the following activities for just a few minutes. Then share ideas with the larger group of 4-6 students.

1) Notice and wonder as much as you can about the problem, focusing on: "Quantities (things I can count or measure)" and "Relationships Between Quantities" (e.g. calculations using operations such as addition, subtraction, multiplication, division; statements about equality, inequality, etc.)
2) Use specific names for the quantities (don't just put the number). Then, if you haven't already, write the values that you know next to the names or descriptions of the quantities, for example in parentheses. Be sure to include units. Example: cost of an apple (50 cents)
3) Write mathematical sentences or equations for the relationships that you understand already, using the names or descriptions of quantities (don't plug in the values yet).
4) Share your lists and sentences with your group. Focus on these questions:
    a. Do I have all the quantities whose value I know?
        i. Do I have specific names for them?
        ii. Do I know the units?
    b. Do I have all the quantities whose values I don't know?
        i. Do I have specific names for them?
        ii. Do I know the units?
    c. Are there relationships that I don't know how to write as a mathematical sentence yet?

Key Outcomes

- Organize and clearly label known information.
- Begin grouping known quantities into relationships.
- Identify which relationships and quantities are known, and which are unknown.

II. Choose Your Approach

Format: Individually and then in pairs or teams.

We present three different ways that mathematicians often build models. You might choose the one that sounds most like what you're thinking, or you might purposely try a new way to see what happens.

Pick one of the following approaches to build your model:

Approach 1: Building Up

Choose this one if:

- You see quantities that you can combine into relationships to form a mathematical model.
- You see ways to break the unknown quantities and relationships into parts that you can figure out.

Building Up: If there are quantities that you have figured out, and you want to make a model with them, you might use the following questions:

- What is the relationship between these quantities?
- What operations would I use to combine these quantities? How would I label the result?

Help if you're stuck: Find a partner and read the quantities and relationships you've combined or created so far. Tell him or her what you would label the results and why. He or she will help you answer, "does that make sense?"

Then tell your partner what relationships you're not sure how to write, or what quantities (be sure to say the name and the units!) you're not sure how to combine. He or she will help you think of new relationships or new combinations to try.

Approach 2: Breaking Down (maybe just for Algebra?)

Choose this approach if: you have a general model (formula) for what you need to figure out in this problem, but don't know the value of some of the quantities

Breaking Down: If you have a general model for what you need to figure out in this problem, but don't know the value of some of the quantities, then you might try to break down the unknown quantities and relationships with the following questions:

- Is this quantity that I don't know the value of related to some other quantity whose value I do know?
- Can I break this relationship down into smaller parts until I get to something I know the value of?

Help if you're stuck: Here are some common mathematical models that mathematicians, scientists, and others use. Do any of these fit your problem?

- Physics
  - Distance = rate * time
  - Height = -4.9(time)^2 + (initial velocity)*(time) + initial height
  - Force = mass * acceleration
- Geometry
  - Circumference = 2*pi*radius
  - Area of a circle = pi * radius^2
  - Area of a rectangle or parallelogram = base * height
  - Area of a triangle = 0.5 * base * height
  - Pythagorean theorem: hypotenuse^2 = leg$_1^2$ * leg$_2^2$
- Business
  - Total cost = cost per item * number of items + initial cost
  - Revenue = cost per item * number of items sold
  - Profit = Revenue – Total Cost
  - Principal $= (Initial\_investment)e^{rate*time}$ (continuously compounded interest)

    $= (Initial\_investment)^{rate*\frac{time\_in\_years}{number\_of\_times\_compounded}}$
- Other
  - Work: $\frac{1}{rate_1} + \frac{1}{rate_2} = 1$
  - Exponential growth: Population = $Initial\_population^{rate*time}$

Approach 3: Guess and Check/Tables (maybe the only one we do for FunPoW?)

Choose this approach if: You're not sure what calculations you can do or you're feeling stuck.

Guess and check and making an organized table can help you write a formula or rules. Those rules can help you calculate the outcomes you're looking for easily, even if you have to do it many times, or you are using hard numbers. Technology like a spreadsheet can really help with this method.

Step 1: Pick a quantity you can make a *guess* for. You'll follow this quantity throughout the whole problem.

Step 2: Do any *calculations* you can do based on the value you guessed. Record the outcomes, and write out the calculation you did as clearly as you can. Label the outcome with a name.

Step 3: Continue to calculate new quantities until you get to a quantity you can *check*. Write what you check using an equals sign.

Step 4: Do as many rounds of guessing, calculating, and checking as you need to until you feel like you can describe step by step what calculations you do and what must be equal at the end.

If you can, use Algebra to write the calculations and check as an equation. Replace the value you guessed with a word or variable.

If not, use words and math symbols to write the rules or steps you do, so that another person could follow your method exactly. You can put the steps in a spreadsheet and it can do the work for you!

Key Outcomes

- Organize the noticed quantities into increasingly comprehensive mathematical models.
- Work step-by-step to continuously refine or expand models.
- Explore multiple methods of approaching a modeling problem.

III. Checking the Model

Format: Students working in pairs.

Multiple strategies are presented below for checking models. Students should complete as many as possible.

Sample Activity:

Compare: With your partner, compare your models. Are they exactly the same? Do they give the same results when you plug in the same values? Is it possible to rewrite the operations so that the models become exactly the same?

If the models are different or give different results, check your work. Is the arithmetic correct? Are the assumptions the same? Do you agree about the relationships?

Try Values: Identify the "constraints" of the problem – the rules or information given in the problem that the result should match. Try plugging some values into your model. Do the outcomes fit the constraints?

Reflect: If the problem had started with different values, could you use your model to work with those different values or would you have to change it? Does your model work for all the different situations of this type of problem?

Key Outcomes:

- Use different strategies to check the models.
- Make sure that models fit all given constraints.
- Evaluate models on the basis of the ease of generalizing them.