The Math Forum: Problems of the Week

Problem Solving and Communication Activity Series

Use Logical Reasoning

Logic is an inherent part of the mathematical problem solving process and was used in some ways through our Activity Series already. However, some problems depend more on logic than on purely mathematical manipulations. Logic can help us find solutions when it looks as if we are unable to solve them based on our equations. Even with problems that are primarily solved through calculations, the questions and techniques of logical reasoning can help us organize and find efficient solution approaches to problems. In this sense *Use Logical Reasoning* is particularly useful in combination with approaches such as "Noticing and Wonderings" or "PoW IQ" from *Understand the Problem* (Rounds 1, 2, or 6) in our Activity Series.

In this document, we focus on using Logic to figure out what is possible and then to narrow down the possibilities until we have our solution.

The activities are written so that you can use them with problems of your choosing.

Problem-Solving Goals

Problem solvers can use logical reasoning to:

- Work from what must be true to figure out what might be true and what can't be true.
- Take what might be true and figure out whether it is true.
- Identify assumptions and figure out what to trust and what to doubt.
- Consider unlikely or unusual ideas that might lead to new solutions.
- Break a problem into sub-problems or cases: "I can try to show that ____, and if it works, I will have solved the problem. If it doesn't, then I will know more about what the answer must be."

Communication Goals

Students who use logical reasoning can use the writing process to:

- Keep track of what they have figured out: ideas that are true, possibilities that might be true, and statements that are definitely not true.
- Keep track of assumptions they trust or doubt.
- Organize trains of logical thinking ("if this is true, then that is true, which means this can't be true…") and make it possible to go back over them to discover mistakes or weak reasoning.

Note: There is an implicit assumption throughout this *Activity Series* that users know that they can also be making use of strategies that have been introduced in prior weeks. In particular, *Understand the Problem* was developed as a set of strategies that is always useful and to some extent assumed to be in use, even when focusing on a new strategy, such as *Use Logical Reasoning*.

Activities

I. Must, Can't, Might

Format: Students working in groups of 2-4.

One important aspect of using logical reasoning is being thorough and careful, not just because we want to avoid mistakes, but because it is also a way to find creative approaches. For this reason, we are going to break the problem solving process into separate steps for both you and your students to get more efficient at this method. When you are doing a problem that is challenging, you will be very happy if you are good at this process.

The first step when reasoning logically is to organize what you know, what information is helpful, and use that information to figure out what else you might know or need to figure out. "Noticing and Wondering" and "PoW IQ"

from the *Understanding the Problem* activities can help generate lists of important mathematical quantities, relationships, and constraints that are given in the problem.

Sample Activity – What Logic Tells Me

Step 1: As you read through the problem for the first time, write the useful information that *must* be true down in a list, a diagram or diagrams, an equation or equations, or some other representation that just has the mathematical information without extra words. Label the list:

- Must be true

Step 2: Write down any obvious *implications* (new ideas that come from the facts you noticed before) from the information you are given, organizing it into three groups:

- What also *must* be true
- *Can't* be true
- *Might* be true

Don't try to solve the problem yet. Just organize all of the information and the most obvious implications.

> *Note*: Logical reasoning takes practice to determine what degree of detail is appropriate. Some people do not write down enough of their ideas, missing important clues, while others write down too much, making it difficult to read and keep track of because they used extra words or thoughts that are not really about the math in the problem. With practice students can learn to write down the appropriate amount for each problem they encounter.

Step 3: Ask each partner how they know what they wrote. Make sure the "true" items are definitely true and each idea or implication is in the correct group.

Key Outcomes

- Organize information given in the problem to make implications more visible.
- Generate possibilities to explore more fully.
- Become aware of assumptions, and what is definitely known versus what is possible.

II. Playing Out Possibilities

Format: Students working individually, then pairing with another group member.

After *Activity I: Must, Can't, Might* you should have some idea of possibilities you can try for the answer. In *Activity II: Playing Out Possibilities* we help you organize and keep track of those explorations. Using logical reasoning to help you work math problems can mean organizing a lot of information. We know from experience playing games like KenKen that it is important to keep track of what you have figured out, and what that reasoning is based on. Then if you make a mistake, you can trace its source without having to start all over; it also makes it a lot easier to explain your thinking and convince yourself and others that you are right.

Sample Activity

Step 1: Each person picks one of the possibilities that *might* be true from *Activity I* and seems like it might lead to the answer, or at least result in good additional information to narrow down the possibilities. Some of the qualities that make for a good possibility to focus on:

- It uses a lot of the information given in the problem.
- You can see how to figure out whether this possibility works or at least how to start figuring it out.
- It has only a small number of different possibilities or cases that you have to consider.

Step 2: Fill in one or more of the following types of statements for the possibility you are exploring:

- If this is true, then that would mean _____ would have to be true.
- In order for that to work, then this would have to be _____.
- I am going to assume that this is _____. Does that cause any problems with the other things I know?

Keep going down the chain of implications as far as you can. Use sentences like the ones above to show the chain of your reasoning: what you hypothesized, what you learned, and why you think it's true.

Step 3: With a partner from your group, take turns telling each other what you are trying. When you share one of your ideas, your partner will ask,

- "How do you know?"
- "What is that fact based on?"

Make a note of the times when you're not sure how you know, or you're not sure if the logic holds up.

Your partner will also help you finde new implication, by helping you make more sentences like:

- If this is true, then…
- In order for this to work, then this has to be…
- Does that cause any problems?...

Keep going until you have each shared and tested everything you discovered.

Step 4: When you find new ideas and facts that you know *must*, *can't*, or *might* be true, add them to your lists from Activity I.

Key Outcomes

- Explore the possibilities (things that may be true) in the problem to fully play out their implications.
- Explore combinations of different things that must be true.
- Develop a method of recording connected logical statements and their justification.
- Justify the solution method to make sure the logic is consistent.

III. Learning from the Logic

Format: Students working in groups of four.

Often solving a problem using logical reasoning involves some leaps of intuition, some wrong turns, or some winding paths that could have been a lot shorter. Sometimes problem-solvers focus on one possibility but would have solved the problem more quickly by putting together new ideas from several of the possibilities.

It can often be difficult to check the result of logical reasoning except by comparing solutions using different methods. It is sometimes said, "The best way to get good at solving problems is to solve a lot of problems." Part of what this means is that problem solvers learn from their previous work. *Activity III* is about writing up your solution so that others can learn from it and compare it to their own solution and everyone can figure out what they now know about solving this kind of problem.

Sample Activity:

Step 1: As a group of four, share the solution paths that you came up with.

- If you did not solve the problem, see if any of the other approaches have ideas you could combine with yours.
- Do any of the solutions contradict each other? See if you can figure out where there is a different assumption or implication that was used. If an error is found, see if you can figure out how it slipped through the logical reasoning process and what someone could do next time to catch it.
- Is there a way to combine the different approaches in your group to make a more efficient solution or to discover new ideas?

Step 2: Individually or with a partner, write up a final solution that is clear and concise. Add a reflection at the end about at least one thing you learned about how to use logic better in solving this type of problem.

Key Outcomes:

- Express your reasoning clearly so that your ideas can be easily understood and examined.
- Compare and learn from each other's solution. Come to a group consensus, as mathematicians do, about valid assumptions and logical reasoning in the problem.