The Math Forum: Problems of the Week

Problem Solving and Communication Activity Series

Work Backwards

Working backwards is a particularly useful problem-solving strategy when you can clearly define the goal or end state of the problem. Maybe you also know the beginning state of the problem, or maybe you know a sequence of operations that were used in the problem. Reversing the operations and working backwards from the goal helps problem-solvers to describe the initial conditions or the path to the beginning state. Working backwards is often applied to logic problems, like the famous one about crossing the river with a cabbage, a goat, and a wolf (http://mathforum.org/library/drmath/view/59217.html), in which you know the goal state (everything on the opposite side of the river), and you know what the legal moves are (rowing one animal across the river without leaving the cabbage with the goat or the wolf with the goat). You can work backwards from the goal, asking yourself, what must have been the last animal rowed across the river? What must have happened just before that?

Another sort of problem that working backwards can be applied to is problems involving operations on a quantity, in which you know the final outcome after all the operations have been applied, and you need to find out the initial quantity. In these sorts of problems, it's useful to play the situation backwards, performing the inverse of each operation on the known, ending quantity until you are left with the initial quantity. This process of inverting operations is very similar to the process of solving an algebraic equation by "undoing" what's been done to the variable.

The activities below help students to clearly define the goal states and beginning states or operations, to imagine reversing the operations, and to check their solutions in a variety of ways.

The activities are written so that you can use them with problems of your choosing.

Problem-Solving Goals

Working backwards can help problem-solvers:

- Discover the mathematical relationships, similar to what you do with the *Guess and Check* strategy without having to guess.
- Find unknown initial conditions when goal states are known.
- Choose the most efficient among multiple paths from initial state to end state.

Communication Goals

Problem-solvers who work backwards might use the writing process to:

- Organize and keep track of complicated sequences.
- Describe the problem context in terms of the quantities and relationships that can be "unwound" or worked backwards.
- Facilitate the transition from verbal descriptions to mathematical representations.
- Make one's thinking available for comparison and testing.

Activities

I. Describing the Problem

Format: students working individually or in pairs.

In order to work backwards, it's important to identify a number or final result at the end of a process that you will "unwind" or "undo" as you work backwards. What are some of the answers or final results that the problem tells you at the end of the situation? What happened, and in what order?

Sample Activity

Work individually or in pairs on the following activities for just a few minutes. You might answer the questions using pictures, a table, arrows, symbols, or words. Try to be as succinct and mathematical as possible. You want to use as few words as possible so that you can focus on the quantities and what was done to them.

1) What do you know about the beginning of the problem? What do you start out with?

2) What do you know about the steps of the problem? List as much as you can in order, from start to finish. What do you know about what steps or operations are allowed and which are not allowed?

3) What numbers or final results do you know that might be a good place to start working backwards? Are there other ways to say the final results? What do you notice about the final stage?

4) Choose what to do next. If you know what the end and the beginning look like, but not the steps in between, choose **Working Backwards Logically.** If you know what the end looks like and what the steps going forwards are, you can choose **Working Backwards Logically** or **Working Backwards by Acting It Out**.

Key Outcomes

- Identify the "goal state" of the problem and any final numbers or results that might be used to work backwards.
- Identify the operations that immediately preceded each state of the problem.
- Writing the steps and operations out clearly so you can undo or unwind the problem.

II. Working Backwards Logically

Format: In pairs.

Working backwards has a lot in common with logical reasoning. We use the language of *must* and *might*, but this time we ask ourselves, "What *must* have happened before?"

Sample Activity:

1) Work with a partner to fill in the blanks in the statement below. Start with describing the ending situation in the first blank.

"If _____ happened, then _____ *must* have happened before."

If you're not sure you can say *must*, then try,

"If _____ happened, then _____ or _____ *might* have happened before."

2) Keep stringing the sentences together for as long as you can, until you get to the beginning situation.

Key Outcomes

- Learn the questions that guide working backwards.
- Solve problems where the end state and beginning state are known.
- Solve the problem.

III. Optional: Working Backwards by Acting It Out

Format: Groups of 4-6, then working in pairs.

Many problem-solvers describe working backwards as playing a movie of the problem backwards in their head. In reverse, giving becomes taking, doubling becomes halving, etc. See what your problem looks like on rewind.

Sample Activity:

1) Act out the story of the problem forwards. Don't worry about numbers for now, just show the steps of the story.

2) Act out the story of the problem in reverse. Show what happens when everything moves backwards.

3) With a partner, take turns describing the action in each step of the backwards movie. Use inverse operations and names of quantities whenever possible.

For example, if one step (going forward) was "Meg gives 3 apples to Sally," then in reverse you might visualize 3 apples moving back from Sally to Meg. The inverse of "give" might be "take" and going backwards it might be "Meg takes 3 apples from Sally". If we wrote it with math symbols we might have going forwards M+3 and S-3 going backwards.

4) When you and your partner agree on a description, write it down.

5) If you haven't already, take a number or result from the end of the problem and use it in your reverse "script" to work backwards and solve the problem.

Key Outcomes

- Change the operations in the problem into their inverses.
- Take an ending number and undo what was done to it to get back to the beginning
- Figure out other ways to describe the problem so that it can be worked backwards more easily.
- Develop a plan by comparing and talking through ideas with a partner.
- Solve the problem.

IV. Checking your Answer

Format: Each pair joins with another pair or two to form groups of 4-6.

Multiple strategies are presented below for checking the answer. Students should complete as many as possible.

Sample Activity:

Comparing: Did each pair have the same steps or different? Do the differences change the answer? Did each pair use the same inverse operations? If not, did this change the answer?

Checking the Constraints: Work forwards from the beginning. Did you start with what you were given, or did you make any assumptions? Did you only do the steps or operations allowed by the problem, or did you do anything that wasn't allowed? At the end, did you get what you were supposed to get? Can you say why it worked, or did you just get lucky?

Generalizing: Did you figure out a pattern or rule you could follow to solve the problem again with different numbers or another version of the problem? Will your solution work for any starting condition, or did you add in extra assumptions?

Using Algebra: Assign variables to the unknowns in your scripts for the problem. Go through the steps of the problem, trying to represent them with algebraic expressions. By the time you get to the end of the problem scenario, you might have algebraic equations that you can plug the ending numbers of the problem into and solve. Compare your methods: Did you get the same answer? Did you discover any errors? Did you discover any new information or relationships?

Key Outcomes:

- Use different strategies to check the solution.
- Retry the problem forwards.